Pre-Midterm Recitation

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1 Definitional Questions

Make sure you know (this is not a complete definition of everything you need to know):

1. How to derive the autocovariance function.
2. How to derive the autocorrelation function.
3. The definition of a weakly stationary model.
4. The definition and interpretation of a statistical model.
5. The definition of identification in a statistical model.
6. The definition of strong stationarity.
7. LLN for iid data
8. CLT for iid data
9. What a Monte Carlo simulation is
10. How to use Monte Carlo simulations with simple models in python
11. Linear Processes
12. Causal Lineary Porcesses
13. Definition and interpretation of IRFs
14. How to attain IRFs in both MA and AR models
15. Definition and intuition of maximum likelihood estimators.
16. Properties of Maximum Likelihood estimators.

Example Questions:

Weak Stationarity

X_t is weakly stationary if:

1) $E[X_t] = \mu < \infty$ and μ does not change over t.

2) $Cov(X_t, X_{t-h}) = E[(X_t - \mu)(X_{t-h} - \mu)]$; $\gamma(h) < \infty \forall h$

and $\gamma(h)$ does not change over t.

e.g. 1) Is the process $X_t = \epsilon_t + \theta\epsilon_{t-1}$ where $\epsilon_t \sim WN(0, \infty)$ weakly stationary?

e.g. 2)a) Choose a value for ϕ_1 and ϕ_2 such that

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

is non-stationary. Show that the process is non-stationary.

e.g. 2)b) Choose a value for ϕ_1 and ϕ_2 such that

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

is weakly stationary. Show that the process is weakly stationary.

Linear Processes Examples

Definition: A Linear Process is a time series process that is a linear weighted sum of white noise shocks:

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \theta_j \epsilon_{t-j}$$

where $\epsilon_j \sim WN(0, \sigma^2)$

Definition: A Causal Linear Process is a linear process such that $\theta_j = 0 \forall j < 0$. That is, a causal linear process takes the form:

$$X_t = \mu + \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j}$$

where $\epsilon_j \sim WN(0, \sigma^2)$

e.g. 1) Is $X_t = \epsilon_{t-1} + \epsilon_t + \epsilon_{t+1}$ a linear process? Is it a causal linear process?

2 Estimation Questions/Examples

2.1 Review of Concepts Used in Estimation

Law of Large Numbers for Time Series
IF:
- Y_t is weakly stationary
- $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$

THEN:

$$\frac{1}{T}\sum_{t=1}^{T} Y_t \to E[Y_t]$$

Examples when this law of large numbers applies:
-Y_t is an MA(q) $\Rightarrow$ LLN applies to Y_t
-Y_t is an MA(∞) with $\sum_{j=0}^{\infty} |\theta(h)| < \infty \Rightarrow$ LLN applies to Y_t
-Y_t is a weakly stationary AR(p) $\Rightarrow$ LLN applies to Y_t

$\star$In short, if Y_t is one of our usual forms and is *weakly stationary*, then the LLN applies. So, for this exam's purposes, just think about weak stationarity. Also, if you know we applied LLN on something in class, you can be pretty sure you can apply it in the exam (e.g. estimating the ACF).

Central Limit Theorem for Time Series
IF:
- Y_t is weakly stationary
- $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$

THEN:

$$\sqrt{T}\sum_{t=1}^{T} Y_t \to E[Y_t] \to N(\mu, \sum_{h=-\infty}^{\infty} \gamma(h))$$

Clarifying Question: Say we have an MA process:

$$X_t = \epsilon_t + \theta\epsilon_{t-1}$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

Under what conditions is the asymptotic variance of $\sqrt{T}\sum_{t=1}^{T} X_t$ (also called the long-run variance) smaller than the variance of X_t?

2.2 LLN applications

Often, questions will ask you to "suggest an estimator" for some parameter of interest. Call this parameter of interest θ. There are generally two types of these questions:
1) Questions where you can apply the LLN directly.
2) Questions where you can write the parameter of interest non-linearly in terms of things you can estimate using LLN.

To answer type 1 questions, find a Y_t such that $E[Y_t] = \theta$. A good answer has the following form: $\frac{1}{T}\sum Y_t = E[Y_t]$ by the LLN. The LLN applies because... $E[Y_t] = \theta$ because...

Example:
Say $X_t = X_{t-1}((\alpha + 1) + \epsilon_t + \theta\epsilon_{t-2}$, where we have data X_t and we assume $\epsilon_t \sim WN(\sigma^2)$.

a. Suggest a consistent estimator for the parameter α, without assuming anything about other parameter values.

Step One: Transform the process to something that has expectation *alpha*.
$X_t = X_{t-1}((\alpha + 1) + \epsilon_t + \theta\epsilon_{t-2}$
$\frac{X_t}{X_{t-1}} = \alpha + 1 + \epsilon_t + \theta\epsilon_{t-2}$
$\frac{X_t}{X_{t-1}} - 1 = \frac{X_t - X_{t-1}}{X_{t-1}} = \alpha + \epsilon_{t-2} + \theta\epsilon_{t-2}$
Note that this is a model with constant growth rate α.

Step Two: Apply law of large numbers.
Define $Y_t = \frac{X_t - X_{t-1}}{X_{t-1}}$
My suggested estimator is the sample mean of Y_t. This estimator is consist, as: $\hat{\alpha}_{\text{proposed}} = \frac{1}{T}\sum Y_t \to E[Y_t]$ by the LLN. The LLN applies because Y_t has the form of an MA(q).
$E[Y_t] = E[\alpha + \epsilon_t + \theta\epsilon_{t-2}]$ by definition.
$= \alpha + E[\epsilon_t] + \theta E[\epsilon_{t-2}]$ by expectation rules
$= \alpha + 0 + \theta 0$ by WN properties
$= \alpha$

b. Assuming you know the value of σ^2, suggest a consistent estimator for θ.

First, I find the expression for σ^2.

Define $Y_t = \frac{X_t - X_{t-1}}{X_{t-1}}$
$Cov(Y_t, Y_{t-2}) = Cov(\alpha + \epsilon_t + \theta\epsilon_{t-2}, \alpha + \epsilon_{t-2} + \theta\epsilon_{t-4})$
$Cov(Y_t, Y_{t-2}) = \theta\sigma^2$
$\Rightarrow \theta = Cov(Y_t, Y_{t-2})\sigma^2$
And we know we can estimate $Cov(Y_t, Y_{t-2})$ consistently using the LLN:

$$\frac{1}{T}\sum_{t=1}^{T}(Y_t - \mu_Y)(Y_{t-2} - \mu_Y) \to Cov(Y_t, Y_{t-2})$$

Therefore, the estimator I propose is:

$$\hat{\theta} = \frac{1}{\sigma^2}\frac{1}{T}\sum_{t=1}^{T}(Y_t - \hat{\alpha})(Y_{t-2} - \hat{\alpha})$$

2.3 Maximum Likelihood Questions

Maximum Likelihood Questions could take two forms:

1. Simple Examples: Derive the likelihood for a simple example (e.g. coin flips)
2. Writing procedure for more complicated likelihoods.

Example:

3 Python Questions

Python Questions will likely be very simple, and will focus on applying techniques we learned. You should know how to answer the question in the homework, how to draw from a normal variable, and how to run for loops. Examples are:

1. Assume the following MA(3) model:

$$X_t = \epsilon_t + \frac{1}{2}\epsilon_{t-1} + \frac{1}{10}\epsilon_{t-2}$$

$$\epsilon_t \sim N(0,1)$$

Say we want to simulate a series with t = 3,...102, then plot it. Can you fill in the blanks in the following code to do so?

```
import numpy
import statistics
import matplotlib.pyplot as plt
```

```
T = range(102)
E = numpy.random.normal(___, __, ___)
-------------------------------------
plt.plot(T, X)
plt.ylabel('X')
plt.xlabel('T')
plt.show()
```

2. Write code to create a 100 by 100 matrix of draws from an iid normal distribution with mean 1 and variance

3. Describe (no need to write code) how you would estimate $\gamma(12) = E[X_t X_{t-12}]$ using Monte Carlo simulations for the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \epsilon_t$$

where $\epsilon_t \sim N(0, 1)$