Recitation Two- Time Series

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February 12, 2019

1 This week

In this recitation, I will talk about a few things:

1) Theorem about $MA(\infty)$ representations of time series.
2) Discussion of LLN for time series v. LLN for iid processes.
3) Simulations of LLN for processes
- IID - Stationary - Non-Stationary

2 Weakly Stationary Processes Reminder

A time series $\{X_t\}_{t\in 1,...T}$ is weakly stationary if
a)$E[X_t] = \mu < \infty$ same for all t
b)$E[(X_t - E[X_t])(X_{t-h} - E[X_{t-h})] = \gamma(h) < \infty$ same for all t

3 $MA(\infty) representations$

Theorem 1. An MA(∞) process is weakly stationary *if* $\sum_{j=0}^{\infty} |\theta_j| < \infty$ Proof:

$$X_t = \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j}$$

$$E[X_t] = \sum_{j=0}^{\infty} \theta_j E[\epsilon_{t-j}] = 0$$

for all t

$$\gamma(0) = Var(X_t) = E[X_t X_t] = \sum_{j=0}^{\infty} \theta_j^2 \sigma^2 < \infty$$

for all t, as long as $\sum_{j=0}^{\infty} \theta_j^2 < \infty$. Now, since $\sum_{j=0}^{\infty} |\theta_j| < \infty \Rightarrow \sum_{j=0}^{\infty} \theta_j^2 < \infty$, the finite variance condition holds when $\sum_{j=0}^{\infty} |\theta_j| < \infty$

The rest of the proof is similar.

Corollary. An MA(q) process is always weakly stationary. Proof:

$$X_t = \theta_0 \epsilon_t + \theta_1 \epsilon_{t-1} + ...\theta_q \epsilon_{t-q}$$

$$E[X_t] = \theta_0 E[\epsilon_t] + \theta_1 E[\epsilon_{t-1}] + ...\theta_g E[\epsilon_{t-q}] = 0$$

for all t

$$\gamma(0) = Var(X_t) = E[X_t X_t] = \theta_0^2 \sigma^2 + \theta_1^2 \sigma^2 + ...\theta_q^2 \sigma^2 < \infty$$

for all t, as long as all $|\theta_j| < \infty$

The rest of the proof is similar.

Theorem 2. (Corollary of Wold's Theorem) Any weakly stationary, purely non-deterministic time series can be represented as an $MA(\infty)$.

Theorem 2.a Any weakly stationary AR, MA, or ARMA process can be represented as an $MA(\infty)$.

Example: AR(1)

Take the model

$$X_t = \phi X_{t-1} + u_t$$

$$u_t \sim WhiteNoise(0, \sigma^2)$$

$$X_t = \phi(\phi X_{t-2} + u_{t-1}) + u_t$$

...

$$X_t = \sum_{j=0}^{\infty} \theta_j u_{t-j} + lim_{j \to \infty} \phi^j X_{t-j}$$

where $\theta_j = \phi^j u_{t-j}$

4 Laws of Large Numbers

Definition 1: A sequence of real-valued random variables $\widehat{\mu}_T$ converges in probability to a constant μ if for any $\epsilon > 0$

$$P(|\widehat{\mu}_T - \mu| > \epsilon) \to 0$$

as $T \to \infty$. We denote convergence in probability as:

$$\widehat{\mu}_T \xrightarrow{p} \mu.$$

Theorem (Law of Large numbers for i.i.d. data): Let $\{X_1, X_2, \ldots\}$ be a collection of i.i.d. random variables each with distribution P and let $\mu \equiv E_P[X_t]$. If $Var(X_t) < \infty$ [or $E[|X_t|] < \infty$] then:

$$\frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow{p} \mu.$$

Theorem (Law of Large numbers for Time Series data; Weak Ergodic Theorem for MA representations): Let $\{X_t\}_{t=1,\ldots,T}$ be time series. If:

1. X_t is weakly stationary
2. $\sum_{j=0}^{\infty} |\theta_j| < \infty$ for the θs in the $MA(\infty)$ representation of the timeseries.

Then:

$$\frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow{p} \mu.$$

In short, weak stationarity allows us to use the LLN for time series! If we do not have weak stationarity, we are in big trouble!

Example: $AR(1)$:

$$X_t = \phi X_{t-1} + \epsilon_t, \epsilon_{t-j} \sim WN(0, \sigma^2)$$

$$Var(X_t) = \phi^2 Var(X_{t-1}) + \sigma^2$$

As $t \to \infty$, $Var(X_t) \to \infty$.

Therefore,

$$P(|\frac{1}{T}\sum_{t=1}^{T} X_t - \mu| > \epsilon) \not\to 0$$