1 Recitation 5; Feb 24,2016 Note

1.1 Finishing Up Discussion of F-tests

For notes on this, see Recitation 4 section of the Weebly website.

1.2 Non-Linear Models Basic Set-Up

Say our model is of the form (That, is, we think the real world looks like):

$$f(Y_i) = \beta_0 + \beta_1 g_1(X_{1,i}, X_{2,i}) + \beta_2 g_2(X_{1,i}, X_{2,i}) + ... + u_i$$

where g_1, g_2,... are non-linear functions of the underlying variables $X_{1,i}, X_{2,i}$ Examples:
1)$g_1(X_{1,i}, X_{2,i}) = X_{1,i}^2$
2)$g_1(X_{1,i}, X_{2,i}) = X_{1,i}X_{2,i}$
1)$g_1(X_{1,i}, X_{2,i}) = ln(X_{1,i})$
Many Others...

How do we estimate these models?:
Step One: Generate our new variables based on the assumed functions. That is, in R, run:
$f < -f(Y)$
$g_1 < -g_1(X_1, X_2)$
$g_2 < -g_2(X_1, X_2)$
...
Step Two: Run the linear regression using those new variables.
i.e. in R, run:
$lm(f \sim g_1 + g_2 + ...)$

1.3 Predictions With Non-Linear Models

Once we have estimated the model, we may want a prediction of $E[Y_i|X_{1,i}, X_{2,i}, ...]$. How do we get this? We estimated from the above procedure the model:

$$E[f(Y_i)|X_{1,i}, X_{2,i}, ...] \equiv f(\hat{Y}) = \hat{\beta_0} + \hat{\beta_1} g_1(X_{1,i}, X_{2,i}) + \hat{\beta_2} g_2(X_{1,i}, X_{2,i}) + ...$$

So to get $E[f(Y_i)|X_{1,i}, X_{2,i}, ...]$, we simply plug in our X's into the above equation! Now, getting $E[Y_i|X_{1,i}, X_{2,i}, ...]$ from this is easy if $f(Y_i) = Y_i$. If it is non-linear, interpretation gets more difficult and we often must take it at a case-by-case basis. The only $f()$ you will encounter in this class is $f(Y) = log(Y)$. IMPORTANT: $E[f(Y_i)|X_{1,i}, X_{2,i}, ...] \neq f(E[Y_i|X_{1,i}, X_{2,i}, ...])$.

1.4 Interpretations of Interaction Models

Often with non-linear OLS models, we are out to answer questions such as: "If X_1 changes by one unit, how much do we expect Y to change by?". In the linear model, the answer was easy, $\hat{\beta}_1$. In the non-linear model, it is more difficult to determine. If $f(Y) = Y$, the easiest way to do it is to directly work out this value by using the formula:

$$E[Y_i|X_{1,i}+1, X_{2,i}, ...] - E[Y_i|X_{1,i}, X_{2,i}, ...]$$

For a interaction model, say

$$Y_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_3 X_{i,2} X_{i,1} + u_i$$

, our estimated model is

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{i,1} + \hat{\beta}_2 X_{i,2} + \hat{\beta}_3 X_{i,2} X_{i,1}$$

The value we want to calculate is:

$$E[\Delta Y_i|\Delta X_{i,1} = 1]$$

$$= E[Y_i|X_{1,i}+1, X_{2,i}, ...] - E[Y_i|X_{1,i}, X_{2,i}, ...]$$

$$= \hat{\beta}_0 + \hat{\beta}_1(X_{i,1}+1) + \hat{\beta}_2 X_{i,2} + \hat{\beta}_3 X_{i,2}(X_{i,1}+1) - [\hat{\beta}_0 + \hat{\beta}_1 X_{i,1} + \hat{\beta}_2 X_{i,2} + \hat{\beta}_3 X_{i,2} X_{i,1}]$$

$$= \hat{\beta}_1 + \hat{\beta}_3 X_{i,2}$$

1.5 Interpretations of Quadratic Models

For a quadratic model, say

$$Y_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_3 X_{i,1}^2 + u_i$$

, our estimated model is

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{i,1} + \hat{\beta}_2 X_{i,2} + \hat{\beta}_3 X_{i,1}^2$$

The value we want to calculate is:

$$E[\Delta Y_i|\Delta X_{i,1} = 1]$$

$$= E[Y_i|X_{1,i}+1, X_{2,i}, ...] - E[Y_i|X_{1,i}, X_{2,i}, ...]$$

$$= \hat{\beta}_0 + \hat{\beta}_1(X_{i,1}+1) + \hat{\beta}_2 X_{i,2} + \hat{\beta}_3(X_{i,1}+1)^2 - [\hat{\beta}_0 + \hat{\beta}_1 X_{i,1} + \hat{\beta}_2 X_{i,2} + \hat{\beta}_3 X_{i,1}^2]$$

$$= \hat{\beta}_1 + \hat{\beta}_3[(X_{i,1}+1)^2 - (X_{i,1})^2]$$

$$= \hat{\beta}_1 + \hat{\beta}_3[2X_{i,1}+1]$$

1.6 Interpretation of Models with logs

First, remember that $log(1+\Delta) \approx \Delta$ if Δ is small. Call this "the amazing log rule" for now.

For these equations, we must change our interpretation a bit.

For a log-linear model, we ask "if $X_{i,1}$ changes by 1 unit, how many percentage points do we expect Y to change by?".
Our simple model is:

$$log(Y_i) = \beta_0 + \beta_1 X_{i,1} + u_i$$

, our estimated model is

$$\hat{log}(Y_i) = \hat{\beta_0} + \hat{\beta_1} X_{i,1}$$

Then, the value we want is:

$$E[\%\Delta Y_i | \Delta X_{i,1} = 1]$$

since $f(Y) \neq Y$, this is not as simple as usual. We can easily calculate

$$E[\Delta log(Y_i) | \Delta X_{i,1} = 1]$$

$$= \hat{\beta_1}$$

Noting that $\Delta log(Y_i) = log(Y_i^1) - log(Y_i^0)$ where the superscripts mean before and after change respectively. And noting that $\Delta Y_i = Y_i^1 - Y_i^0$, we can rewrite the above equation as

$$E[log(Y_i^1) - log(Y_i^0) | \Delta X_{i,1} = 1]) = \hat{\beta_1}$$

By log rules (and dropping the condition just for to make notation easier):

$$E[log(\frac{Y_i^1}{Y_i^0})] = \hat{\beta_1}$$

Adding and subtracting 1:

$$E[log(1 + \frac{Y_i^1}{Y_i^0} - 1)] = \hat{\beta_1}$$

$$E[log(1 + \frac{Y_i^1 - Y_i^0}{Y_i^0})] = \hat{\beta_1}$$

By "the amazing log rule":

$$E[\frac{Y_i^1 - Y_i^0}{Y_i^0}] \approx \hat{\beta_1}$$

$$E[\frac{1}{100}\%\Delta Y_i] \approx \hat{\beta_1}$$

Re-writing this:

$$E[\%\Delta Y_i | \Delta X_{i,1} = 1] \approx 100\hat{\beta_1}$$

For a linear-log model, we ask "if we increase X by 1 percentage point, how much do we expect Y to change by?"

$$Y_i = \beta_0 + \beta_1 log(X_{i,1}) + u_i$$

, our estimated model is

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 log(X_{i,1})$$

Then, the value we want is:

$$E[\Delta Y_i | \% \Delta X_{i,1} = 1]$$

$$= E[Y_i | \frac{101}{100} X_{1,i}] - E[Y_i | X_{1,i}]$$

$$= \hat{\beta}_0 + \hat{\beta}_1 log(\frac{101}{100} X_{i,1}) - [\hat{\beta}_0 + \hat{\beta}_1 log(X_{i,1})]$$

$$= \hat{\beta}_1 [log(\frac{101}{100} X_{i,1}) - log(X_{i,1})]$$

By log rules:

$$= \hat{\beta}_1 [log(\frac{\frac{101}{100} X_{i,1}}{X_i, 1})]$$

$$= \hat{\beta}_1 [log(\frac{101}{100})]$$

$$= \hat{\beta}_1 [log(\frac{1}{100} + 1)]$$

Finally, by the amazing log rule:

$$\approx \frac{1}{100} \hat{\beta}_1$$

Finally, in a log-log model, we ask "if we increase X by 1 percentage point, how many percentage points do we expect Y to change by?"
Our model is

$$log(Y_i) = \beta_0 + \beta_1 log(X_{i,1}) + u_i$$

The value we want is:

$$E[\% \Delta Y_i | \% \Delta X_{i,1} = 1]$$

We transform the left hand side as in the log-linear model and the right hand side as in the linear-log model to get:

$$E[\% \Delta Y_i | \% \Delta X_{i,1} = 1] = 100 \frac{1}{100} \hat{\beta}_1 = \hat{\beta}_1$$

From these values, we can write out our interpretations of $\hat{\beta}_1$ in each of the models. In the log-linear model, the interpretation of $\hat{\beta}_1$ is $\frac{1}{100}$ times "expected percent change in Y due to a 1 unit increase in X". In the linear-log model, the interpretation of $\hat{\beta}_1$ is 100 times "expected change in Y due to a 1 percentage point increase in X". In the log-log model, the interpretation of $\hat{\beta}_1$ is the "expected percent change in Y due to a 1 percentage point increase in X".