Practice Problems - Weeks 1 through 4

With Answers. Important Note: Answers are NOT as full as I would expect in an exam. These are quick answers I wrote for you to check your results. They are not the best answers possible.

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Question One. Statistical Models

Which of the following describe statistical models?

If it is a statistical model, what are the parameters? What are the parameter spaces?

a) $X_t \sim P$ where $P \in${N(0, 2.3), N(1, 2.3), N(2, 2.3), N(3, 2.3)}

YES; μ; {0, 1, 2, 3}

b) $X_t = \beta_0 + \beta_1 X_{t-1} + \epsilon_t$

where $\epsilon_t \sim N(0, \sigma^2)$

YES; $\beta_0, \beta_1, \sigma^2$; $R \times R \times R^+$

c) $E[X_t|Y_t] = 4$

where $Y_t \sim N(2, 4)$

NO, the distribution of the X_t is not defined, just the first moment.

Question Two. Weak Stationarity, Autocovariances.

Which of the following processes are Weakly Stationary? Show a proof.

If they are weakly stationary, derive their autocovariance functions.

a)

$$X_t = .9X_{t-2} + \epsilon_t$$

Where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ iid $\forall t$

b)

$$X_t = .9X_{t-2} + \epsilon_t + .5\epsilon_{t-1}$$

Where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ iid $\forall t$

c)

$$X_t = X_{t-2} + \epsilon_t$$

*I invented these questions. All faults are my own. If you have any concerns, queries, or find something that is incorrect, please email me (ndm2125@columbia.edu)

Where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ iid $\forall t$

d)

$$X_t = t\mu + \epsilon_t + \theta\epsilon_{t-1}$$

Where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ iid $\forall t$

e)

$$X_t = \theta^t exp(\epsilon_t)$$

Where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ iid $\forall t$

a) and **b)** are weakly stationary. The others are not.

Question Two. Stationarity.

a) If a time series process is weakly stationary, is it strongly stationary? Show why using a proof, or why not using a counterexample. [This is a harder question]

No, not necessarily. For example, the third moments could change as t changes.

b) Take the MA(1) process

$$X_t = \theta_0\epsilon_t + \theta_1\epsilon_{t-1}$$

Where $\epsilon_t \sim N(0, \sigma^2)$

Under what values of $\theta_0, \theta_1, \text{and} \sigma^2$ is the process stationary?

All finite values (i.e. $|\theta_0| < \infty$, $|\theta_1| < \infty$, $|\sigma^2| < \infty$)

Question Three. Applications of LLN and CLT.

$\star$ **Note: c) and d) were originally much harder than I had wanted to make them. A more reasonable question would have said added the bold statement below.**

Take the time series process

$$X_t = \phi X_{t-1} + \epsilon_t$$

Where $\epsilon_t \sim N(\mu, \sigma^2)$ iid $\forall t$

a) Suggest a consistent estimator for μ. Which Law of large numbers can you use to prove consistency? Show that its conditions are satisfied.

Answer:

There are many possible answers to this questions. One is as follows:

Step One: Define a consistent estimator for ϕ as $\hat{\phi} = \frac{\hat{Cov}(X_t, X_{t-1})}{\hat{Cov}(X_t, X_t)}$

Step Two: Define $Z_t = X_t - \hat{\phi}X_{t-1}$

Step Three: Our suggested estimator is $\hat{\mu} = \frac{1}{T}\sum Z_t$ We can apply the LLN to this estimator:

$$\hat{\mu} \to E[Z_t] = E[X_t - \phi X_{t-1}] = E[e_t] = \mu$$

This shows that the suggested estimator is consistent if the LLN applies.
To show that the LLN applies to Z_t, show that Z_t is a weakly stationary process.

b) What will be the asymptotic variance of this estimator?

By iid central limit theorem,

$$\sqrt{T}\hat{\mu} \to N(\mu, \sigma^2)$$

Therefore, the asymptotic variance of the estimator
$\hat{\mu}$ is $\frac{1}{T}\sigma^2$

Note: c) and d) were originally much harder than I had wanted to make them. A more reasonable question would have said added the bold statement below.

Now take the time series process

$$X_t = \phi X_{t-1} + \eta_t$$

Where

$$\eta_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$$

And $\epsilon_t \sim N(0, \sigma^2)$ iid $\forall t$
Assume you know the values $\theta = \theta^0$ and $\phi = \phi^0$.

c) Suggest a consistent estimator for μ. Which Law of large numbers can you use to prove consistency? Show that its conditions are satisfied.
Answer:

$$\frac{1}{T}\sum Z_t$$

where $Z_t = X_t - \phi X_{t-1}$ **d)** What will be the asymptotic variance of this estimator?
Answer: By time series central limit theorem,

$$\sqrt{T}\frac{1}{T}\hat{\sum} Z_t \to N(\mu, \sum_{h=-\infty}^{\infty} \gamma(h))$$

Where $\gamma(h)$ is the autocovariance function of Z_t.
$\gamma(0) = Cov(Z_t, Z_t) = Cov(\epsilon_t + \theta\epsilon_{t-1}, \epsilon_t + \theta\epsilon_{t-1}) = (1 + \theta^2)\sigma^2)$
$\gamma(-1) = \gamma(1) = Cov(Z_t, Z_{t-1}) = Cov(\epsilon_t + \theta\epsilon_{t-1}, \epsilon_{t-1} + \theta\epsilon_{t-2}) = (1\theta)\sigma^2)$

Therefore, the asymptotic variance of the estimator
$\sum_{h=-\infty}^{\infty} \gamma_Z(h) = \gamma(-1) + \gamma(0) + \gamma(1)$
$= (1\theta^2 + 2\theta)\sigma^2$

Question Four. Consistency and Unbiasedness.

Suppose that you have an iid sample $X_1, X_2, ..., X_N$ from a $N(\theta, 1)$ model. Consider the following two estimators. The first one in the sample mean:

$$\hat{\theta}_1 = \frac{1}{N} \sum_{i=1}^{N} X_i$$

The second one is the shrinkage estimator:

$$\hat{\theta}_2 \equiv \frac{1}{2}\hat{\theta}_1$$

For each estimator answer the following questions:

a) Is the estimator consistent? Show this.

Answer: Yes for estimator 1. The LLN applies to X_i, as X_i are iid draws.
No for estimator 2. $t\hat{heta}_2 \to E[\frac{1}{2}\hat{\theta}_1] = \frac{1}{2}E[\hat{\theta}_1] = \frac{1}{2}\theta \neq \theta$

b) Is the estimator unbiased? Show this.

Answer: Yes for estimator 1. $E[\frac{1}{N}\sum_{i=1}^{N} X_i] = \frac{1}{N}\sum_{i=1}^{N} E[X_i] = \frac{1}{N}\sum_{i=1}^{N} \theta = \theta$.
No for estimator 2. $E[\hat{\theta}_2] = E[\frac{1}{2}\hat{\theta}_1] = \frac{1}{2}E[\hat{\theta}_1] = \frac{1}{2}\theta \neq \theta$