Math Camp Recitation One

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August 17, 2017

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2 Topic One: Structure and Purpose

Four 1.5 Hour recitations: August 22nd, 24th, 29th, and 31st, plus today, which will be shorter.

In addition, I will likely have a (total of 2 hours) recitation on the 1st and/or the 4th as review sessions for the exam and/or clarification of further topics, depending on demand. Please fill out the short survey on my teaching website if you have a preference.

Recitations are for *you*. I am your research assistant. For each recitation, there will be a survey where you can ask for clarifications on topics, proofs etc., and I will do my best to answer them. This will be the main goal of the recitations, but if not enough questions are asked, I will move on to prepared material. What this planned material also depends on your preferences. Please take the time to fill out the survey.

However, note that I will be focusing my recitation for those who struggle with mathematics, not those who know they want to go into micro theory and already have a masters in mathematics.

3 Topic Two: Mathematical Proofs - an introduction

3.1 Terminology and structure

There are four parts to a mathematical proof :
1) *Axioms/Assumptions* – facts that are assumed to be true.

2) *Theorems* – Previously proven facts
3) Facts derived by rules of inference using Axioms and Theorems
lemma – a theorem that is proven en route to proving a concluding theorem.
4) Concluding *Theorem* – the proven statement
Propositions – less important concluding theorems.
Optional: *Corollaries* – facts that are implied by the concluding theorem.

It is often useful to think at each statement: which part of the proof is this statement?

The basic structure is:
Premises (Axioms and Theorems), then Inference, then Concluding Theorem.

Logic terminology/symbolism:

- $\neg$ - not
- $\wedge$ - and
- $\vee$ -or

3.2 Proving Implications

Many of the statements we want to prove take the form "If P is true, then Q is true."

Terminology:
1) The following are equivalent:

- If P is true, then Q is true
- P *implies* Q
- $P \Rightarrow Q$
- P is a *sufficient condition* for Q and Q is a *necessary condition* for P.
- not(Q)$\Rightarrow$ not(P)

There are four major types of implication proofs:
1)Proof By Construction (*Direct Proofs*)
2)Proof By Contradiction (*Indirect Proofs*)
3)Proof By Induction
4)Proof By Contrapositive

We will go over the structure of each.

PROOF BY CONSTRUCTION:
- This is called a *direct proof*

Want to show: $P \Rightarrow Q$
Premises
Assume P is true.
Inference
Use rules of inference to show that $P \Rightarrow Q$

PROOF BY CONTRADICTION:
- This is called an *indirect proof*
Want to show: $P \Rightarrow Q$
Premises
Assume P is true. Assume not(Q) is true.
Inference
Use rules of inference to show that P and not(Q) is contradictory. I.e. show $\neq$[P and not(Q)].

PROOF BY INDUCTION:
First, Q must take the following form: $Q = \{Q(x) \forall x \in X\}$ where X is an ordered set (x_n is the n^{th} element of X).
Premises
Assume P is true.
Inference
Prove the *basis*: $P \Rightarrow Q(x_1)$
Prove the induction step: $[P \Rightarrow Q(x_n)] \Rightarrow [P \Rightarrow Q(x_{n+1})]$
$[basis \wedge induction\ step] \Rightarrow [P \Rightarrow Q]$

PROOF BY CONTRAPOSITIVE:
Want to show: $P \Rightarrow Q$
Premises
Assume not(Q) is true.
Inference
Use rules of inference to show that $not(Q) \Rightarrow not(P)$
$[not(Q) \Rightarrow not(P)] \Rightarrow [P \Rightarrow Q]$

What if we want to prove that P iff Q? Simply prove using one of the techniques above that P $\Rightarrow$ Q and Q $\Rightarrow$ P.

Note that P is sometimes implicit. For example, if we want to prove that 2x/2=x, what we are really saying is "If x is a member of the real numbers under Euclidean distance then 2x/2=x." P, in short, is anything in your premise.

3.3 Proving Statements with Quantifiers

Quantifiers take three basic forms: 1) Universal quantifiers; For any/For all; $\forall$, 2) Existential quantifiers; There exists; $\exists$, 3) Unique existential quantifiers; There exists exactly one; $\exists!$. The following outlines the general structure for these types of proofs.

UNIVERSAL QUANTIFIERS
We want to show that $P \Rightarrow \forall x \in X, Q(x)$.
Basic structure:
Take any arbitrary $x \in X$.
Show that $P \Rightarrow Q(x)$ using one of the strategies above.

NEGATION OF UNIVERSAL QUANTIFIERS
We want to show that $P \Rightarrow \neg[\forall x \in X, Q(x)]$.
Basic structure:
Find an $x \in X$ such that (assuming P), $\neg Q(x)$.

EXISTENTIAL QUANTIFIERS
We want to show that $P \Rightarrow \exists x \in X | Q(x)$.
We almost always do this by proof in construction and it often takes trial and error. That is, we search for an x in X such that we can prove that $P \Rightarrow Q(x)$ by one of the methods above.

UNIQUE EXISTENTIAL QUANTIFIERS
We want to show that $P \Rightarrow \exists! x \in X | Q(x)$.
Step One: Prove that $P \Rightarrow \exists x \in X | Q(x)$.
Step Two: Prove that this x is unique. We do this by assuming that there are two members of X that satisfy Q(x), then show that these two members are equivalent. That is,
Take $x_1, x_2 \in X | P \Rightarrow Q(x_1) \wedge Q(x_2)$
Prove $[P \Rightarrow Q(x_1) \wedge Q(x_2)] \Rightarrow x_1 = x_2$

Final note of use:

$$\neg[\exists x \in X | Q(x)] \iff [\forall x \in X, \neg Q(x)]$$

$$\neg[\forall x \in X, Q(x)] \iff [\exists x \in X | \neg Q(x)]$$

4 Table On Page 14 Proofs

In $(\mathbb{R}, d_2)$:

$[0, +\infty)$:
Closed:
$[0, +\infty)$ contains all its limit points
$\iff$ $[[0, +\infty)^c$ contains no limit points$]$
So, WTS $x \in [0, +\infty)^c \Rightarrow \exists B_r(x) \cap [0, +\infty)$
Take $x \in [0, +\infty)^c = (-\infty, 0)$
Define $r = abs(\frac{x}{2})$

$B_r(x) \in (-\infty, 0)$
Not Open:
Not[All members of $[0, +\infty)$ are interior points] iff [There exists a member of $[0, +\infty)$ that is not an interior point]
This member is the value 0. For all $r > 0, -\frac{r}{2} \in B_r(0) and - \frac{r}{2} \notin [0, +\infty)$.

$(0, +\infty)$:
Not Closed:
$not[(0, +\infty)$ contains all its limit points

iff [there exists a limit point that is not in$(0, +\infty)$]
This example is 0. 0 is a limit point, as for all $r > 0, \frac{r}{2} \in (0, +\infty)$. Yet, $\frac{r}{2} \notin (0, +\infty)$
Open:
WTS: All members of $(0, +\infty)$ are interior points.
Take an arbitrary $x \in (0, +\infty)$. Define $r = \frac{x}{2}$. $B_r(x) \subset (0, +\infty)$
$\{\frac{1}{n} | n \in \mathcal{N}\}$:
Not Closed:
$\{\frac{1}{n} | n \in \mathcal{N}\}$ contains all its limit points

iff [there exists a limit point that is not in$\{\frac{1}{n} | n \in \mathcal{N}\}$]
This example is 0. 0 is a limit point as for all r¿0, there exists an n such that $(1/n) \in B_r(0)$. Yet, $0 \notin \{\frac{1}{n} | n \in \mathcal{N}\}$
Not Open:
Not[All members of $\{\frac{1}{n} | n \in \mathcal{N}\}$ are interior points] iff [There exists a member of $\{\frac{1}{n} | n \in \mathcal{N}\}$ that is not an interior point]
Indeed, all members are not interior points. Take 1. For all $.5 > r > 0, (1 + \frac{r}{2}) \in B_r(1)$ and $(1 + \frac{r}{2}) \notin [0, +\infty)$.

In $(\mathcal{R}_+, d_2)$:
$[0, +\infty)$:
Open:
[All members of $[0, +\infty)$ are interior points]
Take 0. For all r, $B_r(0) = [0, r) \subset [0, +\infty)$, so 0 is an interior point.
Take an arbitrary $x \in (0, +\infty)$. Define $r = \frac{x}{2}$. $B_r(x) \subset (0, +\infty)$
$\Rightarrow 0 \cup (0, +\infty) = [0, +\infty)$ are interior points.
Closed:
$[[0, +\infty)$ contains all its limit points$]iff[[0, +\infty)^c$ contains no limit points]
So WTS no limit points are in $[0, +\infty)^c$
$[0, +\infty)^c = \emptyset$, which of course, includes no limit points.

$(0, +\infty)$ and $\{\frac{1}{n} | n \in \mathcal{N}\}$:
Same As Before.

In $(\mathcal{R}_{++}, d_2)$:

$(0, +\infty)$:
Closed:
$(0, +\infty)$ contains all its limit points

iff $[(0, +\infty)^c$ contains no limit points]
So WTS no limit points are in $(0, +\infty)^c$
$(0, +\infty)^c = \emptyset$, which of course, includes no limit points.
Open:
WTS: All members of $(0, +\infty)$ are interior points.
Take an arbitrary $x \in (0, +\infty)$. Define $r = \frac{x}{2}$. $B_r(x) \subset (0, +\infty)$

$\{\frac{1}{n}|n \in \mathcal{N}\}$:
Not Open:
Not[All members of $\{\frac{1}{n}|n \in \mathcal{N}\}$ are interior points] iff [There exists a member of $\{\frac{1}{n}|n \in \mathcal{N}\}$ that is not an interior point]
Indeed, all members are not interior points. Take 1. For all $.5 > r > 0, (1 + \frac{r}{2}) \in B_r(1)$ and $(1 + \frac{r}{2}) \notin [0, +\infty)$.
Closed:
$\{\frac{1}{n}|n \in \mathcal{N}\}$ contains all its limit points

iff $[\{\frac{1}{n}|n \in \mathcal{N}\}^c$ contains no limit points]
So WTS no limit points are in $\{\frac{1}{n}|n \in \mathcal{N}\}^c$
Take $x \in \mathcal{R}_{++} \cap \{\frac{1}{n}|n \in \mathcal{N}\}$
Define $r = min_{n \in \mathcal{N}}\{|\frac{1}{n} - x|\}$
r¿0, as for all x¿0, we can find two members of $\mathcal{N}, n_1$ and n_2 such that $x \in (\frac{1}{n_1}, \frac{1}{n_2})$.
$B_r(x) \subset \mathcal{N}\}^c \Rightarrow$ x is not a limit point.