Math Camp Recitation Three: Lineariarizing

Nathaniel Mark

August 24, 2017

1 Why do we want to linearize things?

In modern (DSGE) macroeconomics, our models often take the form of:
Representative consumer(s)/worker(s) maximizing utility functions under dynamic constraints.
Representative firm(s) maximizing profit under production constraints.
Equilibrium concepts.
Finally, there are generally stochastic shocks that enter the model somewhere (e.g. productivity shocks, taste shocks, liquidity shocks).

These models must have non-linearities to catch some stylized facts about the economy: e.g. decreasing returns to scale, diminishing marginal utility. So to attain predictions from these models (after we have parameter values already), we have three options:
1) Simulate shocks, solve the non-linear system at each step, and calculate summary statistics (such as means) over the simulations.
2) Approximate the system by linearizing it, and conduct the simulation exercises discussed in option 1.
3) If it is simple enough, approximate the system by linearizing it, then solve it by hand! With this solution, we can observe its predictions directly!

The third option is obviously the nicest, and you will do this many times in your macro class. The second is much nicer than the 1st, as it is much faster. And the first would take forever.

2 Key Tool for Linearizing: The Taylor Theorem

As stated in your notes, the Taylor theorem states that:

For $f : A \to \mathbb{R}$, under some regularity conditions [1], $\forall x, x_0 \in A, \exists \tilde{x} \in (x, x_0)$ s.t.

[1] Domain is closed, f is C^{n-1} (its n-1st derivative is continuous), and f^n exists for all members in the interior of the domain

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f^{(2)}(x_0)}{2}(x-x_0)^2+...+\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}+\frac{f^{(n)}(\tilde{x})}{n!}(x-x_0)^n$$

Or equivalently,

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + R_n(\tilde{x})$$

where $R_n(\tilde{x}) = \frac{f^{(n)}(\tilde{x})}{n!}(x-x_0)^n$.

$\sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$ is called the Taylor Approximation of $f(x)$ around x_0.

3 Thinking about Taylor Approximations and Linearizing

In the notes, it is proven that $R_n(\tilde{x})$ is $o((x-x_0)^n)$.

Just a refresher (or if you do not know this, you should learn it):

''Big O" around 0:
$f(x) = O(g(x)$
IFF
$\exists M, \bar{x}$ s.t.
$|f(x)| < M|g(x)| \forall x s.t. |x| < |\bar{x}|$
That is, in some area around 0, f(x) is bounded by g(x) multiplied by some number. For this to be true, f(x) must converge at to 0 at the same speed or faster than g(x) converges to 0.

''Little O" around 0:
If $\exists \bar{x} s.t. \forall x > \bar{x}, g(x) > 0$,
$f(x) = o(g(x))$
IFF
$lim_{x \to 0} \frac{f(x)}{g(x)} = 0$

That is, f(x) converges to 0 faster than g(x) as x goes to 0.

If we a linearizing, we set n=2. So, our approximation of f(x) around x_0 is:

$$f(x_0) + f'(x_0)(x-x_0)$$

And our error of this approximation (i.e. $f(x) - Approx(f(x))$) is:

$$\frac{f''(\tilde{x})}{2}(x-x_0)^2$$

for some $\tilde{x} \in (x, x_0)$
We know from above that this converges to 0 faster than $(x - x_0)^2$. Is this a good approximation then? Not necessarily. This only says it is a good approximation super super close to x_0. What if we are not that close? Then this does not really tell us anything.

Indeed, what we have to do when assessing whether the Taylor approximation is a good approximation is to think: in the area that I am talking about, does the function look quadratic? If so, it will be a good approximation for points in that area. If not, it will not be a good approximation.

4 Why log-linearizing?

So, thinking about this: does the Cobb-Douglas production function's first derivative look approximately quadratic? No! but is does look like a log function!

This is the basis of why we log linearize. In macroeconomics, it makes more sense to talk about percent deviations than in absolute deviations. For the most part, this is because we assume utility functions and production functions that rely on percent deviations.

5 Three Methods of Log-Linearizing

Before we start going over methodology, here is a little terminology (for economists).

x_0 - The ''steady state." This is what we approximate around. The name comes from macro.

$\hat{x}$ - Percent deviation from the steady state.
Notes: $\hat{x} \equiv \frac{x - x_0}{x_0} \approx log(\frac{x}{x_0})$ [if $x - x_0$ is close to 0]

5.1 Method One

Step One: Take Logs
Step Two: Take (Linear) Taylor Approximations (of both sides)
Step Three: Transform so that all values are in terms of ''steady states" and ''percent deviations from steady states. (Note: we often use the fact that we know the equation holds with equality in the steady state and subtract this steady state equation.)

Cobb-Douglas Example:

$$Y = L^\alpha K^\beta$$

Taking Logs:

$$log(Y) = \alpha log(L) + \beta log(K)$$

Taylor Approximating:

$$log(Y_0) + \frac{Y - Y_0}{Y_0} \approx \alpha(log(L_0) + \frac{L - L_0}{L_0}) + \beta(log(K0) + \frac{K - K_0}{K_0})$$

Subtracting $log(Y_0) = \alpha log(L_0) + \beta log(K_0)$ from the equation.

$$\frac{Y - Y_0}{Y_0} \approx \alpha(\frac{L - L_0}{L_0}) + \beta(\frac{K - K_0}{K_0})$$

By definition:

$$\hat{Y} \approx \alpha\hat{L} + \beta\hat{K}$$

[2]

5.2 Method Two

The second method relies on the knowledge that $\hat{x} \approx log(\frac{x}{x_0})$ (by a taylor approximation of the log) and the note then that $x \approx x_0 exp(\hat{x})$.

Step One: For all variables, x, we want to log-linearize over, substitute $x_0 exp(\hat{x})$ for x.
Step Two: Take (Linear) Taylor Approximation of the new function with respect to $\hat{x}$'s deviations from 0.
Step Three: Simplify. (Note: we usually use the fact that we know the equation holds with equality in the steady state and divide by this steady state equation.)

Cobb-Douglas Example:

$$Y = L^\alpha K^\beta$$

Replacing:

$$Y_0 e^{\hat{Y}} \approx (L_0 e^{\hat{L}})^\alpha (K_0 e^{\hat{K}})^\beta$$

Taylor Approximating around 0:

$$Y_0 + Y_0(\hat{Y}) \approx L_0^\alpha K_0^\beta + \alpha L_0^\alpha K_0^\beta(\hat{L}) + \beta K_0^\beta L_0^\alpha(\hat{K})$$

[2] I used https://www3.nd.edu/ esims1/log$_linearization_sp12.pdf formethodoneandhttp$: $//www.columbia.edu/$ $nc2371/teaching/R9.pdf formethodstwoandthree.$

Now, divide everything by $Y_0 = L_0^\alpha K_0^\beta$:

$$1 + \hat{Y} \approx 1 + \alpha\hat{L} + \beta\hat{K}$$

Which is the same as in method 1!

I think this method is unwieldy, but some people like it because of its natural relationship with elasticities.

5.3 Method Three

The final method is just using tricks and simply apply the log-linearizing function to the entire equation.

Step 1: Apply the log deviation function to both sides.
Step 2: Iteratively apply the log deviation function rules until the function is written only as log deviations of variables and steady states.

To do so, we define the log deviation function $\widehat{\ }: \mathbb{R} \to \mathbb{R}$, as:

$$\widehat{f(x)} = log(\frac{f(x)}{f(x_0)})$$

This function can be applied to any function of variables.

Our list of tricks are:
Note that these are written in terms of one variable, but we can always replace the variable with a function of variables and the variable's steady state with a function's steady states.

$$\widehat{x+y} = \frac{x_0}{x_0+y_0}\hat{x} + \frac{y_0}{x_0+y_0}\hat{y}$$

$$\widehat{x^\alpha} = \alpha\hat{x}$$

$$xy = \hat{x} + \hat{y}$$

$$\hat{c} = 0$$

if c is constant.

This generally the easier way to log-linearize.

Cobb-Douglas Example:

$$Y = L^\alpha K^\beta$$

Apply the log deviation function to both sides:

$$\hat{Y} = \widehat{L^\alpha K^\beta}$$

Use tools:

$$\hat{Y} = \widehat{L^\alpha} + \widehat{K^\beta}$$

$$\hat{Y} = \alpha\hat{L} + \beta\hat{K}$$

Much easier!

6 Examples

In the following examples, I just use method 3.

Euler Equation (vars are c's and r)

$$(\frac{c_{t+1}}{c_t})^\sigma = \beta(1+r)$$

log-linearize:

$$\widehat{(\frac{c_{t+1}}{c_t})^\sigma} = \beta\widehat{(1+r)}$$

$$\sigma\widehat{(\frac{c_{t+1}}{c_t})} = \widehat{\beta} + \widehat{(1+r)}$$

$$\sigma\widehat{c_{t+1}} - \sigma\widehat{c_t} = \widehat{(1+r)}$$

$$\sigma\widehat{c_{t+1}} - \sigma\widehat{c_t} = \frac{1}{1+r_0}\hat{1} + \frac{r_0}{1+r_0}\hat{r}$$

$$\sigma\widehat{c_{t+1}} - \sigma\widehat{c_t} = \frac{r_0}{1+r_0}\hat{r}$$

Note: if you want to see how this is done using method 1, see page 5 of https://www3.nd.edu/ esims1/log_linearization_sp12.pdf.

Resource constraint:

All Latin letters are variables. All Greek letters are constants.

$$c + k' - (1-\delta)k = ak^\alpha l^\beta$$

$$c + k' \widehat{- (1-\delta)}k = a\widehat{k^\alpha} l^\beta$$

$$c + k' \widehat{- (1-\delta)}k = \widehat{a} + \alpha\widehat{k} + \beta\widehat{l}$$

$$\frac{c_0}{c_0 + k_0' + (1-\delta)k_0}\widehat{c} + \frac{k_0'}{c_0 + k_0' + (1-\delta)k_0}\widehat{k'} - \frac{(1-\delta)k_0}{c_0 + k_0' + (1-\delta)k_0}\widehat{(1-\delta)k} = \widehat{a} + \alpha\widehat{k} + \beta\widehat{l}$$

$$\frac{c_0}{c_0 + k_0' + (1-\delta)k_0}\widehat{c} + \frac{k_0'}{c_0 + k_0' + (1-\delta)k_0}\widehat{k'} - \frac{(1-\delta)k_0}{c_0 + k_0' + (1-\delta)k_0}\widehat{k} = \widehat{a} + \alpha\widehat{k} + \beta\widehat{l}$$