Math Camp Recitation Two

Nathaniel Mark

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1 Test Information

The test will be a similar format to previous years. It will be on Tuesday 9/5, 6:10pm-8:30pm in 405 International Affairs Building. If you cannot make this time, please email Amy Devine (aed2152@columbia.edu) as soon as possible. Tentatively, the exam will have a TRUE/FALSE section, as well as (likely) three longer-form questions. The TRUE/FALSE section will make up 25% of the grade, as will each of the longer questions.

2 Visualizing Different Types of Sets

Say S is a subset of X in the metric space (X,d).

2.1 Open and closed sets

$$int(S) = \{x : \exists r > 0 | B_r(x) \subset S, x \in X\}$$

That is, the interior of a set are the set of points in the metric space where there exists an open ball around that point that is entirely inside the set S. These points are, in a sense, ‘’sufficiently far for not(S).”

S is an open set iff $S = int(S)$. That is, S is an open set iff all points in S are interior points.

$$lp(S) = \{x : \forall r > 0, (B_r(x)/x) \cap S \neq \emptyset, x \in X\}$$

That is, the set of all limit points is the set of all points in the metric space where any open ball around that point, but not including the point, is going to touch points in the set. These points are, in a sense, in a sense, ‘’sufficiently close to the interior of S.”

S is closed iff $lp(S) \subset S$. That is, S is a closed set iff it contains all of its limit points.

Easier way to define a closed set (in my point of view):

$$cl(S) = \{x : \forall r > 0, (B_r(x)) \cap S \neq \emptyset, x \in X\}$$

That is, the *closure* of S is the set of all points in the metric space where any open ball around that point (including that point) is going to touch points in the set. These points are, in a sense, in a sense, ''sufficiently close to S.''

S is closed iff $cl(S) = S$. That is, S is a closed set iff is equal to its closure.

Proposition: $cl(S) = lp(S) \cup S$. That is, The closure of a set is the union of S itself and the set of all limit points.
Proof: $\Leftarrow$: Take $x \in lp(S)$. $x \in cl(S)$ trivially as $(B_r(x)/x) \cap S \neq \emptyset \Rightarrow (B_r(x)) \cap S \neq \emptyset$.
Take $x \in S$. $\forall r > 0, B_r(x) \cap S \supset x \neq \emptyset \Rightarrow x \in cl(S)$.
$\Rightarrow$: Take $x \in (lp(S) \cup S)^c$. $\forall r > 0, (B_r(x)/x) \cap S = \emptyset \wedge x \cap S = \emptyset \Rightarrow (B_r(x)) \cap S = \emptyset$.

Finally, we define the boundary of the set S as the subset of the closure of S that is not in the interior of S. That is,

$$Boundary(S) \equiv \partial S = cl(S)/int(S)$$

Note that the boundary of S is also the boundary of S^c.
Proof: $\partial S = cl(S)/int(S) = cl(S) \cap (int(S))^c = (int(S^c))^c \cap (cl(S^c)) = \partial S^c$

Now, lets do some (as Stephane Dupraz calls them) Potato drawings to solidify this understanding.

2.2 Bounded, Complete, and Compact Sets

A set is bounded if there exists an open ball of finite radius that covers it. That is, S is *bounded* if $\exists r < \infty, x \in X | S \subseteq B_r(x)$.

A metric space (X,d) is bounded if $\exists r < \infty, x \in X | X \subseteq B_r(x)$.

A metric space (X,d) is *complete* if all Cauchy sequences converge to a point in X.

Officially, only metric spaces can be complete. But, we can talk about subsets of a metric space being complete in the sense that ''S in complete'' if (S,d) is a complete metric space.

Finally, a subset S in (X,d) is *compact* if for every open cover of S in X, there exists a finite sub-cover.

Completeness and compactness are tough concepts to visualize as separate from closedness. The following provides some intuition that will hopefully help.

Complete seems very similar to closed right? Indeed, they are related.

List of implications:

1) If S is complete (in X), then S is closed.
2) If S is compact (in X), then S is complete.
3) If S is compact (in X), then it is closed. (implied already)
4) If X is complete, then S is complete iff S is closed.
5) If X is compact, then S is compact iff S is closed.
6) If S in compact, then it is bounded.
7) If X is closed and bounded, then X is compact.

One good explanation of compactness I've seen is the following: Compact sets have a finiteness property in the sense that no matter what kind of (open) shape you think about, a finite number of those open shapes can cover the set.

If you only really think in $\mathbb{R}^n$, then the differences between these three are fairly nuanced and generally have to with the form of the metric space we are working in and boundedness constraints.

For example:

$(1, 5]$ is closed, but not complete or compact in $((1, \infty), d_2)$.

Proof of nots: the Cauchy sequence $1 + \frac{1}{n}$ does not converge in $((1, \infty), d_2)$.

not(complete)⇒not(compact) by contrapositive.

$[1, \infty)$ is closed and complete, but not compact in $((1, \infty), d_2)$.

Proof of not: not(bounded)⇒not(compact) by contrapositive.

3 Limits and Continuity

See attached.

4 Basics/Tricks of Linear Algebra

The following are some useful things to know for proofs in linear algebra.

Transposes:

A'B'=BA

Or, equivalently,

A'B=B'A

Trace (all except the last are obvious):
If A and B are both nxn,
tr(A)=tr(A')
tr(cA)=ctr(A)
tr(AB)=tr(BA)

Inverse:
If A and C are non-singular ($det \neq 0$),
$(A^{-1})' = (A')^{-1}$
$(AB)^{-1} = B^{-1}A^{-1}$
And maybe useful: $(A+C)^{-1} = A^{-1}(A^{-1}+C^{-1})^{-1}C^{-1}$

Determinants:
det(A)=det(A')
det(cA)=c^ndet(A)
det(AB)=det(A)det(B)
$det(A^{-1}) = det(A)^{-1}$

Eigenvalues:
Say $\lambda_1, ..., \lambda_n$ are the eigenvalues of A. Then,
$det(A) = \prod_{i=1}^{n} \lambda_i$
$tr(A) = \sum_{i=1}^{n} \lambda_i$

Positive Definiteness:
A positive definite iff A is symetric and all its eigenvalues are positive. A positive definite $\Rightarrow$ A is nonsingular. A positive definite $\Rightarrow$ A^{-1} positive definite

5 Rank of AB

Someone asked why $rank(AB) \leq min\{rank(A), rank(B)\}$ and not =.
Linear combination of vector subspaces can have lower basis than the original vector subspace, they just cannot have larger. For example, see

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

The first matrix has rank 2, the second has rank 1, but the first times the second has rank 0.