Recitation - Sep 25

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October 2, 2017

1 Optimization

In this class, we do a lot of optimization problems. That is, agents maximize some kind of function with constraints attached.

$$max_{x1,x2}\pi(x1,x2)$$

$$s.t. f_1(x1,x2) \leq a$$

$$s.t. f_2(x1,x2) \leq b$$

How do we solve these problems? In our class, we generally assume everything the following process will find the maximum. There are some cavaets, which you will learn if you go further in economics.

STEP ONE: Prove which constraints are *binding* and which constraints are *slack*.

Binding Constraints are constraints that hold with equality at the optimum. That is, if x_1^*, x_2^* are a solution to the problem, then $f_1(x_1^*, x_2^*) = a$

Slack Constraints are constraints that hold with strcit inequality at the optimum. That is, if x_1^*, x_2^* are a solution to the problem, then $f_1(x_1^*, x_2^*) < a$

STEP TWO: rewrite the problem using only the *binding constraints* holding with equality. e.g.

$$max_{x1,x2}\pi(x1,x2)$$

$$s.t. f_1(x1,x2) = a$$

STEP THREE: Plug the binding constraints into the objective function:

$$max_{x1,x2}O(x1,x2)$$

STEP FOUR: Take the first derivative with respect to all of the variables the agent is choosing and set each of them equal to zero. These equation are called the *First Order Conditions*. e.g.

$$\frac{\partial O(x1,x2)}{\partial x1} = 0$$
$$\frac{\partial O(x1,x2)}{\partial x2} = 0$$

STEP FIVE: Solve the system of equations for x1 and x2. These are your solutions.

Very simply Example:

$$max_{x_1,x_2}x_2 - x_1^2$$

$$s.t.x_2 \leq -1$$

$$x_2 \leq -2$$

First, we prove that the first constraint is binding at the optimum by contradiction. That is, we assume that it is not binding and show that that means it is not an optimum. Then, we prove that the second constraint does not bind at the optimum. We can use the fact that we already proved that the first constraint does bind at the optimum to prove this. Our rewritten problem is then

$$max_{x_1,x_2}x_2 - x_1^2$$

$$s.t.x_2 = -1$$

Plugging in the constraint:

$$max_{x_2} - 1 - x_1^2$$

FOCs:

$$-2x_1 = 0 \implies x_1 = 0$$

So the solution is $x_1 = 0, x_2 = -1$.

2 Monopoly

So far, we have talked about models for how monopolies act in four different markets.

2.1 No Price Discrimination

In the no price discrimination model, the monopoly only knows the demand curve. They cannot charge a different price to every consumer based on the consumer's preferences, they cannot offer different prices for different quantities, and they cannot segment the market.

How should a monopoly set their price to maximize profits? Their problem is:

$$max_p \pi(p) = D(p)p - c(D(p))$$

FOC:

$$D'(p)p + D(p) = c'(D(p))D'(p)$$

$$p + D(p)/D'(p) = c'(D(p))$$

$$p + q\frac{\partial p}{\partial q} = c'(q)$$

How should a monopoly set their quantity to maximize profits? Their problem is:

$$max_q \pi(q) = qP(q) - c(q)$$

FOC:

$$P(q) + qP'(q) = c'(q)$$

$$p + q\frac{\partial p}{\partial q} = c'(q)$$

We end up with the same thing? Why?

Really, we are solving

$$max_p \pi(p) = qp - c(q)$$

$$s.t. q = D(p)$$

So, by the constraint, choosing a q and choosing a p are the same. Note that this will not always be true in every model!!

Now, people often write this in terms of price elasticity. Remember,:

Price elasticity is denoted ϵ. It is defined as $\epsilon = \frac{P\partial Q}{Q\partial P}$.

Taking our MC=MR equation above:

$$p + q\frac{\partial p}{\partial q}\frac{p}{p} = c'(q)$$

$$p(1+\frac{q}{p}\frac{\partial p}{\partial q}) = c'(q)$$

$$p(1+\frac{1}{\epsilon}) = c'(q)$$

$$p = \frac{1}{1+\frac{1}{\epsilon}}c'(q)$$

Thinking about elasticity: Say we have a demand curve: $P(q) = 10 - \alpha q$. Then, plugging in the derivative of this demand curve and P itself into the elasticity equation, we have $\epsilon = \frac{10-\alpha q}{-\alpha q} = -\frac{10}{\alpha q} + 1$. At any given q, a higher alpha (steeper demand) decreases ϵ (becomes more inelastic) while a lower alpha (less steep demand) increases ϵ (becomes more elastic). For any linear elasticity, increasing q decreases ϵ (becomes more inelastic). WARNING: saying "elasticity increases" can be very confusing! Be clear!

2.2 First degree Price Discrimination

In a first degree price discrimination model, the firm can perfectly predict the willingness to pay of each consumer. For example, say the market is 10 people, 2 of which are willing to pay $5 for a cup of coffee, 2 are willing to pay $4, ...The firm knows who is who. Note that these markets don't really exist. Amazon/ other online sellers using your past purchase history was the closest. Past Auctions can also reveal your preferences.

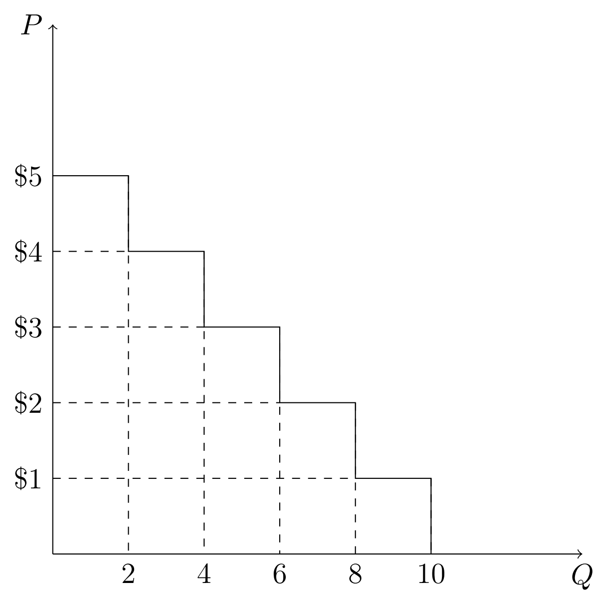

Figure 2.1 – Example of demand

- If the monopoly observes this demand knowing exactly the willingness to pay of each consumer, and there is no resale, how would it set the prices ? It will sell the first 2 units at \$5, the next 2 at \$4, the next 2 at \$3, the next 2 at \$2.

- The marginal revenue curve is the same as the demand curve!
- First degree price discrimination is socially optimal. There is no deadweight loss!
- Why do we have step demand like this instead of a smooth curve ? If instead we have the next 2 consumers value the good at \$4.99, and the next 2 at \$ 4.98 ... demand will be smoother.

2.3 Third degree Price Discrimination

A market with third degree price discrimination is one where the *market can be segmented* and *it is not easy for people to resell items across market segments.* For example, selling groceries in different neighborhoods, airline tickets to businesses/vacationers, etc.

Example

Demand for organic milk in Harlem:

$$Q_1 = 10 - P_1$$

Demand for organic milk in UWS:

$$Q_2 = 10 - \frac{P_2}{2}$$

Total Cost:

$$TC(Q) = 2 + Q^2$$

The monopolist will maximize:

$$\max_{P_1, P_2} Q_1 P_1 + Q_2 P_2 - TC(Q_1 + Q_2)$$

How do we solve these problems?

STEP ONE: Generally, the problem is easier if we transform the demand to $P(Q)$ and the maximization problem to Q. Why?

So, solve the demand functions for P's and rewrite the problem:

$$\max_{Q_1, Q_2} \quad \underbrace{Q_1(10 - Q_1)}_{\text{Revenue from market 1}} + \underbrace{Q_2(20 - 2Q_2)}_{\text{Revenue from market 2}} - 2 - (Q_1 + Q_2)^2$$

STEP TWO: Take FOCs

Using FOC

$$MR_1(Q_1) = MC(Q_1 + Q_2)$$

$$MR_2(Q_2) = MC(Q_1 + Q_2)$$

$$10 - 2Q_1 = 2(Q_1 + Q_2)$$

$$20 - 4Q_2 = 2(Q_1 + Q_2)$$

STEP THREE: Solve the system:
Set FOC 1 =FOC 2

$$10 - 2Q_1 = 20 - 4Q_2$$

$$Q_1 = 2Q_2 - 5$$

Plug into FOC 2:

$$20 - 4Q_2 = 2(3Q_2 - 5)$$

$$30 = 10Q_2$$

$$Q_2^* = 3, Q_1^* = 1$$

$P_1 = 9, P_2 = 14$

We could think of this as solving in two steps: first, finding the optimal $Q = q_1 + q_2$, then finding the optimal q_1 and q_2 to maximize profit given Q.

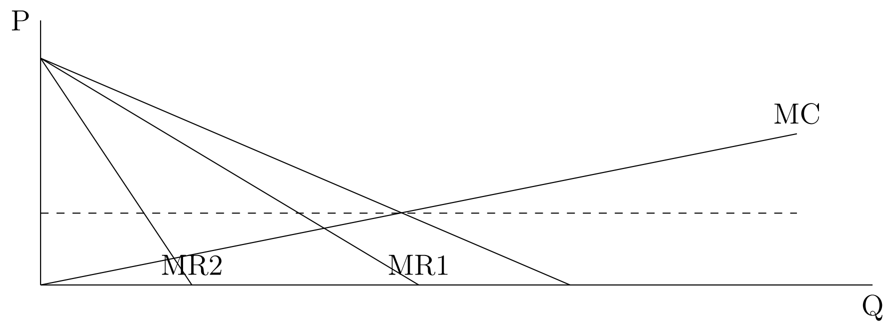

Figure 2.2 – How would the monopolist set the prices for these 2 markets ?

2.4 Second degree

- When the monopolist cannot segment the market (cannot observe the type of the consumer)
- Goal: Design a menu to make consumer reveal themselves! This is usually called an adverse selection problem.

Example

The monopolist knows that there is 2 types of consumers in the market: one prefers coffee a lot and one doesn't prefer as much.
For type 1: the utility from drinking q oz of coffee and paying t is $\theta_1 q - t$
For type 2: $\theta_2 q - t$
If $\theta_1 < \theta_2$, which type prefers coffee more ? Assume that the fraction of type 1 is λ, what's the fraction of type 2 ?
Since there are only 2 types, the monopolist would want to design a menu with 2 choices (q_1, t_1) and (q_2, t_2) to maximize his profits.

$$\max_{q_1,t_1,q_2,t_2} \lambda(t_1 - C(q_1)) + (1-\lambda)(t_2 - C(q_2))$$

There are four constraints:

$$\theta_1 q_1 - t_1 \geq 0$$

This is called individual rationality for low type (IR -L), The left hand side is a low type's consumer's utility from purchasing the the option designed for his type. This condition ensures that the low type consumer would purchase instead of staying out of the market.

$$\theta_2 q_2 - t_2 \geq 0$$

Similarly, individual rationality for high type (IR - H)

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2$$

This is incentive compatibility condition for low type (IC-L). The left hand side is the low type's utility from purchasing the option designed for his type (θ_1, q_1), right hand side is the low type's utility from purchasing the option designed for the other type (θ_2, q_2). Remember that the goal is you want to design a menu so that just by looking at which option the consumer picks, you immediately know his type!

$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1$$

This is incentive compatibility condition for high type (IC-H). Similar intuition as above.
Results when λ is sufficiently big

- IR-L binds: the low type gets zero surplus
- IC-H binds: the high type is indifferent between buying (θ_1, q_1) and (θ_2, q_2)
- the high type retains some surplus (unlike the low type)
- When there are more than two types, the above intuition still holds (Example)
- what happens if λ is small ?