IO Recitation One: Game Theory Basics *

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1 What is Game Theory?

Game theory is a framework to analyze how rational actors will interact in an environment where one actor's actions affect the other actors' actions. We can think of games as loosely being made up of five aspects:
1)) The set of *Players* in the game.
2) The *Actions* available to each player in the game.
3) The *Information* each player has in the game when they choose their action.
4) *Payoffs*: A measure of how happy each actor is, given their choice of action and other players' choices of actions. Usually this is in terms of profit (for firms) or utility (for consumers).
5) *Equilibrium concept*: to find solutions to the game, we think about different ways that players can act rationally.

2 Notation

i *Players*: $I = \{1, 2, ..., N\}$

ii *Actions*: The set of actions player i can choose from is denoted A^i. Each player chooses an action from that set. The chosen action is denoted a^i.

   The *Outcome* of the game is the list of actions chosen by each player $(a^1, a^2, ...a^N)$

iii The payoff, $\pi^i(a^1, a^2, ...a^N)$, depends on all of those actions.

   Often, the payoff function will be written as $\pi^i(a^i, a^{-i})$. Here a^i denotes the action taken by player i and a^{-i} denotes a list of all the actions taken by other players.

Let's take an example: say there are two players, robber 1 and robber 2, who rob a bank together. They are both brought in for questioning and have already been charged with a carrying an illegal firearm. If they both confess, they both go to jail for 5 years.

*Much thanks to Anh Nguyen for previous work on this note.

If one confesses and the other does not, the confessor does not go to jail at all but the non-confessor goes to jail for 10 years. If neither confess, they both go to jail for one year for the firearms charge.

i *Players*: $I = \{1, 2\}$

ii *Actions*: Possible Actions are "Confess" and "Don't Confess". (A^i={Confess, Don't Confess} for both i)

iii Outcomes of this game: (Confess, Don't Confess), (Don't Confess, Confess), (Confess, Confess), (Don't Confess, Don't Confess).

Just to make sure you understand notation, note that the outcome (Confess, Don't Confess) means that Robber 1 chose Confess and Robber 2 chose Don't Confess.

iv The payoff, $\pi^i(a^1, a^2)$, depends on all of those actions:
π^1(Confess, Don't Confess) = 0
π^1(Confess, Confess) = -5
π^1(Don't Confess, Confess) = -10
π^1(Don't Confess, Don't Confess) = -1

Same for player 2.

3 Game Representations

There are two ways to write down games: In *Normal Form* and in *Extensive Form*

3.1 Normal Form

Normal form is most useful for games where all players choose their actions at the same time and there are only two players (These both can be augmented).
In this game form, the game is written down in a table format:

		Player 2		
		a^1*option*1	...	a^1*optionK*
Player 1	a^2*option*1	$(\pi^1(a^1, a^2), \pi^2(a^1, a^2))$	...	$(\pi^1(a^1, a^2), \pi^2(a^1, a^2))$
	...	...	$\ddots$	...
	a^2*option*2	$(\pi^1(a^1, a^2), \pi^2(a^1, a^2))$	...	$(\pi^1(a^1, a^2), \pi^2(a^1, a^2))$

Our Example:

		Robber 2	
		Confess	Don't Confess
Robber 1	Confess	(-5,-5)	(0,-10)
	Don't Confess	(-10, 0)	(-1,-1)

3.2 Extensive Form Game

Extensive form game is more useful if there is timing involved in the game. For example, if one player can observe the other player's action before they choose their action, then extensive form is much easier to deal with. For example, say in our original game that robber 1 is first given the opportunity confess, then police officers tell robber 2 what action robber 1 took before they give her the opportunity to confess. We want to be able to include this "information acquisition" into the game. We then write the game in extensive form.

See Attached Handwritten Notes.

The last thing to note about extensive game form is the concept of a subgame. A *subgame* consists of a **single** decision node and **all** its successors in the original game. Each subgame could be played as its own game! Figure one has a circle over every possible subgame in two example games.

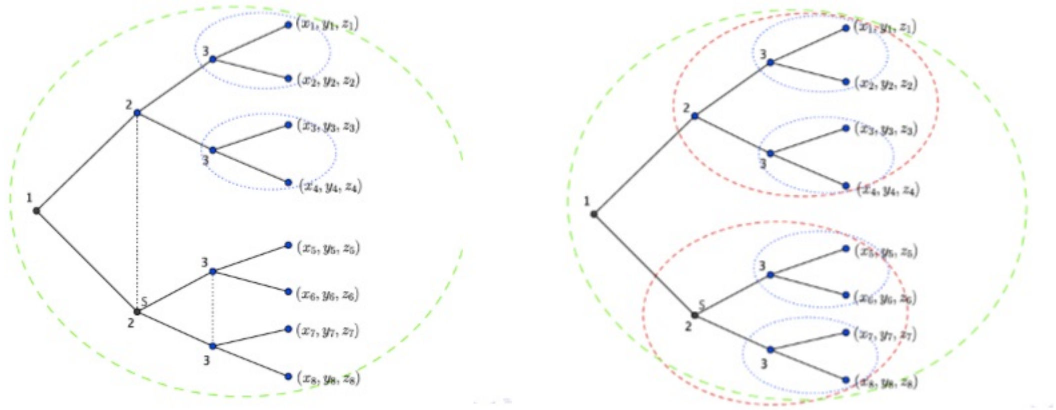

Figure 1: Examples of Subgames

4 Equilibrium Concepts

In a sense, Equilibrium concepts are different ways of answering the question "which *outcome* will occur if all players are rational in a certain way?"

4.1 Equilibrium in dominant actions

- An equilibrium in dominant actions is one in which all players play the action that is **always** best. This is called the *dominant action.*
- This is a pretty logical way to think about how players are rational right? Yes, but there is a problem that most games don't have this equilibrium.
- We solve these problems in two steps:
  Step One: For each player, find all dominant actions.
  Step Two: The outcomes where all actions are dominant actions are "Equilibrium in dominant actions"

4.2 Nash Equilibrium

- Informally: Every player is taking his best action given what the other players are doing. Equivalently, no player can deviate to take another action and be better off as a result.

- This is the concept that we use most in IO.
- We solve these problems in two steps:
  Step One: For each player and all possible opposing player's strategies, find "best response" actions.
  Step Two: The outcomes where all actions are best responses are "Nash Equilibrium"

  Lets solve some examples of normal form games for Nash and Dominant equilibria:

		Robber B: Confess	Robber B: Don't Confess
Robber A	Confess	-5,-5	0,-10
	Don't Confess	-10,0	-1,-1

(a) 1 Nash Equilibrium

		Nate's GF: NYC	Nate's GF: Westchester
Nate	NYC	10,5	0,0
	Westchester	0,0	5,10

(b) 2 Nash Equilibriums

		Player 2: Heads	Player 2: Tails
Player 1	Heads	-1,1	1,-1
	Tails	1,-1	-1,1

(c) No Pure Strategy Nash Equilibrium

4.3 Subgame Nash Equilibrium

- Now, adding a timing aspect, what would be expect players to rationally do? Perhaps they should assume that all players will be rational (in the Nash sense) in the future. Then, with this assumption, they chose the best response action given those expected future actions. This idea is formalized in the definition of a subgame nash equilibirum.
- A s*ubgame perfect Nash Equilibrium* is a strategy profile which constitutes a Nash equilibrium in every subgame.

- We solve these problems iteratively:
Step One: Start with the subgames closest to the terminal nodes. Find the Nash equilibrium of that game.
Step Two: Act as if the starting node of those solved subgames are now terminal nodes with payoffs corresponding to their Nash Equilibria. Step Three: Repeat steps 1 and 2 until all nodes have an assigned action. This set of actions corresponds with a subgame nash equibirium.

- Remark 1:If any subgame has more than one Nash Equilibrium, you have to do step 2 with each Nash Equilibrium outcome.

- Remark 2: All SPNE are Nash equilibrium. But a Nash equilibrium might not be a SPNE

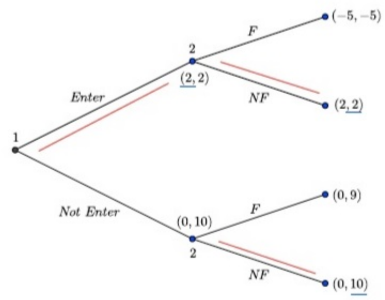

Figure 2: Example of SPNE

5 Repeated Game

A repeated game is a dynamic game in which the same static game is played at every stage. A repeated game might have a finite or infinite number of repetitions.

5.1 Finitely Repeated Game

- In a finitely repeated game, if the stage game has a **unique** Nash Equilibrium, then the finitely repeated game has a unique SPNE in which the Nash equilibrium is played at every stage game.

- What if the stage game doesn't have a unique equilibrium: can solve the game backward.

- Example: consider playing the (modified) prisoner's dilemma game twice
Each player i has a strategy of the following form

$$\{A_1^i, \big(\big(A_2^i|[C, C], A_2^i|[C, NC], A_2^i|[NC, C], A_2^i[NC, NC]\big)\big)\}$$

	Player 2 C	Player 2 NC
Player 1 C	1,1	5,0
Player 1 NC	0,5	4,4

– A_1^i: Player i's action in the first stage
– $A_2^i|[C,C]$: Player i's action in the second stage conditional on $[C,C]$ was played in the first stage
– The unique SPNE of this game is $[C,C]$

5.2 Infinitely Repeated Game

- How to check if a strategy profile is a SPNE ? Check if anyone wants to deviate!
- Also need to care about discount factor δ: how you value the future vs present.
- Example: consider the following stage game being infinitely repeated

	Player 2 C	Player 2 NC
Player 1 C	1,1	5,0
Player 1 NC	0,5	4,4

Is a strategy profile in which (i) if the previous stage game outcome is (NC,NC), then each player plays NC; (ii) otherwise, each player plays C. [Answer] Given that my player plays NC throughout, should I deviate [1]?
If I stick to the strategy and plays NC throughout, I get:

$$4+4\delta+4\delta^2+...=\frac{4}{1-\delta}$$

If I deviate, I get 5 for the current period, but my opponent will play C in the next period. Given that my opponent is playing C next period, I'm playing C in the next period as well (Check to see that (C,C) is a Nash equilibrium). Subsequently, he will keep playing C (remember that I'm assuming that my opponent is sticking to the strategy). Therefore, my profit is:

$$5+1\delta+1\delta^2+...=5+\frac{\delta}{1-\delta}$$

[1] note that this game is symmetric, so only need to check one player. If the game is not symmetric, have to check for both players

That is, I'm getting 5 in the current period, but 1 in every subsequent period (and future profits are discounted by δ).
When will I not deviate?

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}$$

This is easy to solve and the above condition is equivalent to:

$$\delta \geq \frac{1}{4}$$

Remark: The more patient I am, the less likely that I will deviate. I'll be punished in the future if I deviate!