This note is meant to help those who want a clearer proof of how we get classical measurement error bias. This is not material you have to know in this much detail.

1 Classical Measurement Error

The set-up:
True Model:

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \epsilon_i$$

We run the model

$$Y_i = \gamma_0 + \gamma_1 \tilde{X}_{1,i} + u_i$$

where

$$\tilde{X}_{1,i} = X_{1,i} + v_i$$

and v_i is a mean 0 random variable that is independent to $X_{1,i}$. That is, the data we have are simply the true data with a little independent randomness added on.
What do we want to show? We want to find the asymptotic bias of the $\hat{\gamma}_i$ estimator as an estimate for β_1. Now, we know that in the simple OLS model,

$$\hat{\gamma}_i \to \frac{Cov(\tilde{X}_i, Y_i)}{Var(\tilde{X}_i)} as n \to \infty$$

Plugging in the True Model to get rid of Y_i:

$$\hat{\gamma}_i \to \frac{Cov(\tilde{X}_i, \beta_0 + \beta_1 X_{1,i} + \epsilon_i)}{Var(\tilde{X}_i)}$$

Plugging in the definition of $\tilde{X}_{1,i}$ from above:

$$\hat{\gamma}_i \to \frac{Cov(X_{1,i} + v_i, \beta_0 + \beta_1 X_{1,i} + \epsilon_i)}{Var(\tilde{X}_i)}$$

Since β_0 is a constant:

$$\hat{\gamma}_i \to \frac{Cov(X_{1,i} + v_i, \beta_1 X_{1,i} + \epsilon_i)}{Var(\tilde{X}_i)}$$

Since v_i and ϵ_i are independent from $X_{1,i}$:

$$\hat{\gamma}_i \to \frac{Cov(X_{1,i}, \beta_1 X_{1,i})}{Var(\tilde{X}_i)}$$

Since β_1 is a constant:

$$\hat{\gamma}_i \to \beta_1 \frac{Cov(X_{1,i}, X_{1,i})}{Var(\tilde{X}_i)}$$

Since $Cov(X_{1,i}, X_{1,i}) = Var(X_{1,i})$,

$$\hat{\gamma}_i \to \beta_1 \frac{Var(X_{1,i})}{Var(\tilde{X}_i)}$$

By definition of $\tilde{X}_i$):

$$\hat{\gamma}_i \to \beta_1 \frac{Var(X_{1,i})}{Var(X_{1,i} + v_i)}$$

$$\hat{\gamma}_i \to \beta_1 \frac{Var(X_{1,i})}{Var(X_{1,i}) + Var(v_i) + 2Cov(X_{1,i}, v_i)}$$

And finally, by independence of v_i and $X_{1,i}$,$Cov(X_{1,i}, v_i)$=0:

$$\hat{\gamma}_i \to \beta_1 \frac{Var(X_{1,i})}{Var(X_{1,i}) + Var(v_i)}$$

So, $\hat{\gamma}_i$ will underestimate β_1